Bases

We start up with Linear independence:
The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent (LI) if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

implies that

$$a_1 = a_2 = a_3 = \cdots = a_n = 0,$$

or, in other words, that the set of vectors can only create the zero vector via linear combination if the combination is the trivial (all zero) one. The alternate state of affairs, Linear Dependence (LD) means that the vectors CAN be linearly combined to the zero vector WITHOUT resorting to multiplying everything by zero. These are fairly simple definition, with a lot of implications.

Theorem:
A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if and only if one of its elements can be expressed as a linear combination of the others.

Proof.
First, one direction. Start with ONE of the vectors being expressible as a linear combination of the others. We'll assume the vector in question is $\mathbf{v}_1$, for clarity (it can be any of them, and the order they're in is arbitrary, etc).

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_3$$

can be rewritten by subtracting $\mathbf{v}_1$ from each side for

$$\mathbf{0} = (-1)\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

which makes the set linearly dependent (since the coefficients are not all equal to zero).

The other way is equally easy. We start with

$$\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

There's at least one a_k in there that's not equal to zero. This time we'll rearrange the vectors so that $a_n \neq 0$. So, subtract $a_n\mathbf{v}_n$ from each side for

$$-a_n\mathbf{v}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{n-1}\mathbf{v}_{n-1}$$

and divide each side by $-a_n$ for a final expression

$$\mathbf{v}_n = -\frac{a_1}{a_n} - \frac{a_2}{a_n}\mathbf{v}_2 - \cdots - \frac{a_{n-1}}{a_n}\mathbf{v}_{n-1},$$

and done. $\blacksquare$

Property: If, possibly with some rearrangement, you can get

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

$$\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \operatorname{Span}\{\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}.$$

Proving this is not too hard, it'll probably be on an assignment (maybe with a specific number of vectors, or something of the sort). So, those terms in a set are unnecessary for the span, they are redundant. If the set is not Linearly Independent, then it's got unnecessary terms (with respect to the span), which can be removed due to this result.

Definition: A minimal spanning set for V is a set of vectors that spans V but will not do so if ANY ONE of its vectors is removed from the set.

Algorithm (to make a minimal spanning set):

1. Start with a vector space V and a spanning set of V, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
2. Get the full solution set to the problem
$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$
3. If the solution is NOT just $a_1 = a_2 = \cdots = a_n = 0$:
   (a) Take a vector $\mathbf{v}_k$ that had a variable (i.e., NOT JUST ZERO) solution to the problem. Remove it from the set (so we get a new n value, etc).
   (b) Go Back to Step 2.
4. If the solution IS just zeroes, you are done.

So, that procedure (which is not fun to use, I can tell you) can take any FINITE SIZED spanning set and break it down until it is linearly independent, and therefore the smallest it can be while still spanning the set.

Example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$$

is a linearly dependent set. Setting up the LI/LD problem leads to the equations

$$a = 0 \qquad b + c + 2d = 0 \qquad a + 2b + c + 3d = 0.$$

Ignore a, since it's zero, then subtract the second equation twice from the third for $-c - d = 0$. Add that to the second equation for $b + d = 0$. This gives $c = -d$ and $b = -d$, d variable. This DOES NOT make the fourth element the variable equation, it makes the second, third, and fourth variable. The first equation is NOT expressible in terms of the other, we can't get rid of it without interfering in the span. We can remove the second to fourth, but the fourth one is the most complicated, so take

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

So, we can take a spanning set and break it down into a linearly independent set without losing the spanning set property.

Property: If a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly independent and the vector $\mathbf{x}$ does not belong to the span of the set then

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{x}\}$$

is a linearly independent set.

Proof.
This one is fairly easy. Take

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m + b\mathbf{x} = \mathbf{0},$$

which is equivalent to

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m = -b\mathbf{x}.$$

This is not possible for $b \neq 0$, so we get

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m = \mathbf{0},$$

which has only the trivial solution (the $\mathbf{v}$ are LI). So, we only get the trivial solution, the set is linearly independent. ■

Definition: a set is *maximal linearly independent*in V if you can not include any additional vector from V in the set without making it linearly dependent.

Algorithm (to make a maximal linearly independent set):

1. Start with a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
2. Consider the subspace $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
3. If there is a vector $\mathbf{x} \in V$ that is NOT in W:
   (a) include it in the set, so $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{x}\}$.
   (b) Go back to step 2.
4. If there are no vectors in V outside W, then $W = V$ and you are done.

Example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is linearly independent. It is not, however, a spanning set for $\mathbb{R}^3$, it actually represents the plane $-x - y + z = 0$. It's easy, in this case, to find a vector not in that set. How about we just expand it to

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now to define what we are actually creating with these two processes.

Definition: A *Basis* for the vector space V is a linearly independent spanning set of V.

Basic Examples:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for } \mathbb{R}^3 \qquad \{x^2, x, 1\} \text{ for } \mathcal{P}_2$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ for } \mathcal{M}_{2\times 2}$$

and so on.

Less Basic Examples:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for } \mathbb{R}^3 \qquad \{x^2 - 1, x^2 + x, x - 1\} \text{ for } \mathcal{P}_2.$$

Now to look into the size of bases.

Dimension

The Fundamental Theorem (of vector spaces?)

Take a vector space

$$V = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$$

and the linearly independent set

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_k\} \in V.$$

(Notice that the $\mathbf{x}$ go to k, the $\mathbf{v}$ go to n.)

We always get $k \leq n$.

This is not particularly easy to prove, but hopefully will make some sense. This one is a proof by contradiction. The idea is you start with the *opposite* of what you want, then prove it to be impossible. So, we start by assuming that $k = n + 1$ then show that this makes the set of $\mathbf{x}$ vectors linearly dependent.

Proof.

The first step is to write $\mathbf{x}_1$ as a linear combination of the $\mathbf{v}$ vectors:

$$\mathbf{x}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

This is possible since the $\mathbf{v}$ vectors span all of V and $\mathbf{x}_1 \in V$. Also, $\mathbf{x}_1 \neq \mathbf{0}$, since the set of them is not linearly dependent, so at least ONE of the a_k values is $\neq 0$. So, rearrange the $\mathbf{v}$ so that we get

$$\mathbf{x}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \quad a_1 \neq 0$$

so we can get

$$\mathbf{v}_1 = -\frac{1}{a_1}\mathbf{x}_1 + \frac{a_2}{a_1}\mathbf{v}_2 + \cdots + \frac{a_n}{a_1}\mathbf{v}_n.$$

Since $\mathbf{v}_1$ is a linear combination of $\mathbf{x}_1$ and $\{\mathbf{v}_2, \dots \mathbf{v}_n\}$ we can say that

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \text{Span}\{\mathbf{x}_1, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \text{Span}\{\mathbf{x}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}.$$

We do that again:

$$\mathbf{x}_2 = b_1\mathbf{x}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n.$$

Again, we can be sure that there is some a value there that is $\neq 0$. The $\mathbf{x}$ are linearly independent, so the $\mathbf{v}$ vectors will be necessary. Reorder so that the $a_2 \neq 0$. So:

$$\mathbf{v}_2 = \frac{b_1}{a_2}\mathbf{x}_1 - \frac{1}{a_2}\mathbf{x}_2 + \frac{a_3}{a_2}\mathbf{v}_3 + \cdots + \frac{a_n}{a_2}\mathbf{v}_n.$$

We do the same thing yet again, leading eventually to

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}.$$

Repeat as needed until we get

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}.$$

But wait, that means

$$\mathbf{x}_{n+1} \in V = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_x, \dots, \mathbf{x}_n\},$$

so

$$\mathbf{x}_{n+1} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n$$

which contradicts the $\mathbf{x}$ vectors being linearly independent. $\blacksquare$

So, what does this mean? It puts a limit on the size of linearly independent sets in the vectors space V equal to the size of the smallest spanning set. It also means:

Corollary: If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ are bases of the vector space V then $n = m$ (so they have to be the same size).

This is easy to prove: each set is both linearly independent and a spanning set for V, so we apply the Fundamental Theorem both ways for $n \leq m$ and $m \leq n$, so $m = n$.

This may seem trivial, but it means that the *size of a basis* in V is *fixed.* It's a fundamental property of the vector space. We call it dimension:

Definition: If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ (n vectors total) is *any* basis for vector space V, then the dimension of V, $\text{Dim}(V)$, is equal to n.

Definition: If V has no finite basis then $\text{Dim}(V) = \infty$.

Notice that if we can find a linearly sized spanning set then we can get a basis using the algorithm from earlier.

Theorem:

Take any vector space V. One of these (only) is true:

1. V has a finite basis, of some size $n \in \mathbb{N}$, so $\mathrm{Dim}(V) = n$. Every independent set in V has size $\leq n$. All spanning sets for V have size $\geq n$.

2. V does not have a finite basis, so $\mathrm{Dim}(V) = \infty$. Independent sets in V can be of ANY size. V has no finite spanning set.

Examples:

Here's the basics:

$$\mathbb{R}^n, \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}, \quad n \text{ of those, so } \mathrm{Dim}(\mathbb{R}^n) = n.$$

$$\mathcal{P}_n, \quad \left\{ x^n, x^{n-1}, x^{n-2}, \ldots, x, 1 \right\}, \quad n+1 \text{ of them, so } \mathrm{Dim}(\mathcal{P}_n) = n+1.$$

Here's one that's more lengthy to write: $\mathcal{M}_{n\times m}$ has dimension $n \times m$, as you would expect. So, $\mathcal{M}_{2\times 2}$ has dimension 4.

The symmetric subspace of $\mathcal{M}_{2\times 2}$ is three dimensional:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Planes of $\mathbb{R}^3$ that are subspaces are two dimensional, etc etc.

The vector space $\mathcal{P}$ is infinite dimensional. How do we establish that? By proving that ANY finite set is NOT a spanning set.

Take a set in $\mathcal{P}$:

$$\{p_1(x), p_2(x), \ldots, p_n(x)\}.$$

Each of those is a polynomial, each is in $\mathcal{P}$, so each has a finite order (top power of x). Give those the numbers

$$\{Y_1, Y_2, \ldots, Y_n\}.$$

Next, take the biggest one: $Y = \max\{Y_1, Y_2, \ldots, Y_n\}$. Now, is the polynomial $x^{Y+1} \in \mathcal{P}$ within $\mathrm{Span}\{p_1, p_2, \ldots, p_n\}$? No. It's not a spanning set. Done.

Now some equivalent properties:

Theorem:

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \in V$ is either ALL of the following or NONE:

1. a basis of V

2. a minimal spanning set of V

3. a maximal LI set in V

4. an independent set in V, where $\mathrm{Dim}(V) = n$

5. a spanning set for V, with $\mathrm{Dim}(V) = n$

Exercises:

Section 5.2:
3bd, 4, 5, 11, 15
22 ($\mathbb{R}\mathbf{v}$ means the set of all $a\mathbf{v}$ with $a \in \mathbb{R}$)

Section 4.3:
4bd